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STABILITY OF STEADY-STATE SHEAR JET FLOWS OF AN IDEAL FLUID WITH A FREE BOUNDARY IN AN AZIMUTHAL MAGNETIC FIELD AGAINST SMALL LONG-WAVE PERTURBATIONS

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The problem of the linear stability of steady-state axisymmetric shear jet flows of a perfectly conducting inviscid incompressible fluid with a free surface in an azimuthal magnetic field is studied. The necessary and sufficient condition for the stability of these flows against small axisymmetric long-wave perturbations of special form is obtained by the direct Lyapunov method. It is shown that if this stability condition is not satisfied, the steady-state flows considered are unstable to arbitrary small axisymmetric long-wave perturbations. A priori exponential estimates are obtained for the growth of small perturbations. Examples are given of the steady-state flows and small perturbations imposed on them which evolve in time according to the estimates obtained.

Key words: *jet shear flows, long-wave approximation, stability, direct Lyapunov method.*

1. Formulation of Exact Problem. An infinitely long cylindrical jet of a perfectly conducting inviscid incompressible fluid in unbounded space is studied. It is assumed that an azimuthal magnetic field is frozen in jet material and a longitudinal constant electric current flows over the jet free surface, producing a quasi-steady-state azimuthal magnetic field in the unbounded space enclosing the jet. In addition, it is assumed that the examined magnetohydrodynamic jet flows of an ideal fluid are axisymmetric and the azimuthal component of the velocity field is identically equal to zero. The surface tension on the free boundary of the conducting jet is ignored.

By virtue of these assumptions, the equations of one-fluid ideal magnetohydrodynamics [1] take the form

$$\rho\Big(\frac{\partial v_1}{\partial t^*} + v_1\,\frac{\partial v_1}{\partial r^*} + v_3\,\frac{\partial v_1}{\partial z^*}\Big) + \frac{H_2^2}{4\pi r^*} = -\frac{\partial P_*}{\partial r^*}, \qquad \rho\Big(\frac{\partial v_3}{\partial t^*} + v_1\,\frac{\partial v_3}{\partial r^*} + v_3\,\frac{\partial v_3}{\partial z^*}\Big) = -\frac{\partial P_*}{\partial z^*},$$
$$\frac{\partial H_2}{\partial t^*} + v_1\,\frac{\partial H_2}{\partial r^*} + v_3\,\frac{\partial H_2}{\partial z^*} - \frac{v_1 H_2}{r^*} = 0, \qquad \frac{1}{r^*}\,\frac{\partial(v_1 r^*)}{\partial r^*} + \frac{\partial v_3}{\partial z^*} = 0, \tag{1.1}$$

where $\rho \equiv \text{const}$ is the density field, v_1 and v_3 are the radial and axial components of the velocity field, H_2 is the azimuthal component of the magnetic field inside the jet, P is the pressure field, $P_* \equiv P + H_2^2/(8\pi)$ is the modified pressure field, t^* is time, and r^* and z^* are cylindrical coordinates. It is assumed that the axis z^* of the cylindrical coordinate system coincides with the axis of the conducting jet.

Ignoring displacement current, the azimuthal component H_2^* of the magnetic field outside the jet is given by the formula

$$H_2^* = 2J/r^* \tag{1.2}$$

($J \equiv \text{const}$ is the magnitude of the surface longitudinal constant electric current).

On the axis of the conducting jet and its free boundary, the following boundary conditions are specified:

$$v_1 = 0, \quad |H_2/r^*| < +\infty \qquad (r^* = 0),$$
$$P_* = \frac{H_2^{*2}}{8\pi}, \quad v_1 = \frac{\partial r_1}{\partial t^*} + v_3\,\frac{\partial r_1}{\partial z^*} \qquad [r^* = r_1(t^*, z^*)]. \tag{1.3}$$

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The initial data for the first three relations of system (1.1) and the last boundary condition (1.3) are specified as

$$v_1(0,r^*,z^*) = v_{10}(r^*,z^*), \qquad v_3(0,r^*,z^*) = v_{30}(r^*,z^*),$$
$$H_2(0,r^*,z^*) = H_{20}(r^*,z^*), \qquad r_1(0,z^*) = r_{10}(z^*), \tag{1.4}$$

and it is required that the functions v_{10}, v_{30}, H_{20}, and r_{10} be consistent with the fourth equation of system (1.1) and the first three relations in (1.3).

Next, in the mixed problem (1.1)–(1.4), we pass to the long-wave approximation preceded by a nondimensionalizing procedure. The nondimensionalizing parameters are as follows: L is the characteristic spatial scale of variation in the hydrodynamic and magnetic fields along the axis z^*, v_0 is the characteristic velocity of the fluid, and r_0 is the characteristic radius of the jet. These parameters are used to introduce the nondimensional quantities t, η, z, q, w, p_*, h, and æ, so that the following relations hold:

$$t^* = tL/v_0, \quad r^{*2} = \eta L^2\delta^2, \quad z^* = zL, \quad 2v_1r^* = qv_0L\delta^2,$$

$$v_3 = wv_0, \quad P_* = p_*\rho v_0^2, \quad H_2 = hr^*\sqrt{4\pi\rho v_0^2}\,/(L\delta), \quad H_2^*r^* = æ\sqrt{4\pi\rho v_0^2}\,L\delta.$$

Here $\delta = r_0/L \ll 1$ is the nondimensional characteristic radius of the conducting jet.

After nondimensionalizing using the above relations, system (1.1) is written as

$$\delta^2(q_t + qq_\eta - q^2/(2\eta) + wq_z)/2 + \eta h^2 = -2\eta p_{*\eta},$$
$$w_t + qw_\eta + ww_z = -p_{*z}, \qquad h_t + qh_\eta + wh_z = 0, \qquad q_\eta + w_z = 0. \tag{1.5}$$

Using relation (1.2), boundary conditions (1.3) are brought to the form

$$q = 0, \quad |h| < +\infty \qquad (\eta = 0),$$
$$p_* = æ^2/(2\eta_1), \quad q = \eta_{1t} + w\eta_{1z} \qquad (\eta = \eta_1(t,z)), \tag{1.6}$$

where $æ \equiv J/(r_0\sqrt{\pi\rho v_0^2}\,) = \text{const}$.

The initial data (1.4) are written as

$$q(0,\eta,z) = q_0(\eta,z), \qquad w(0,\eta,z) = w_0(\eta,z), \qquad h(0,\eta,z) = h_0(\eta,z), \qquad \eta_1(0,z) = \eta_{10}(z). \tag{1.7}$$

If in the first of Eqs. (1.5) terms proportional to δ^2 are omitted and if the expression for the function $q(0,\eta,z)$ is eliminated from relations (1.7), the initial-boundary-value problem (1.5)–(1.7) reduces to a form that corresponds to the long-wave approximation. This representation of problem (1.5)–(1.7) cannot be considered final because it can further be simplified by replacing the Eulerian independent variables t, z, and η by mixed Eulerian–Lagrangian independent variables t', z', and ν [2]. By analogy with [3], this substitution is performed by the formulas

$$t = t', \qquad z = z', \qquad \eta = R(t',z',\nu), \qquad \nu \in [0,1].$$

It is assumed that the function R satisfies the equation

$$q = R_{t'} + wR_{z'} \tag{1.8}$$

and the boundary conditions

$$R(t',z',0) = 0, \qquad R(t',z',1) = \eta_1(t',z'). \tag{1.9}$$

The essence of this substitution of independent variables is that by means of the Lagrangian variable ν, it is possible to enumerate the trajectories of fluid particle in the jet. In addition, from the definition of the function R [see (1.8) and (1.9)] it follows that boundary conditions (1.6) are satisfied for the function q automatically. Finally (and this is the main thing), according to the substitution of independent variables, the unknown free surface of the conducting jet $\eta = \eta_1$ becomes the well-known fixed boundary $\nu = 1$.

Thus, in the new mixed Eulerian–Lagrangian independent variables (if we neglect terms with δ^2), system (1.5) becomes

$$R_\nu h^2 = -2p_{*\nu}, \qquad R_\nu(w_t + ww_z) = -R_\nu p_{*z} + R_zp_{*\nu},$$
$$h_t + wh_z = 0, \qquad q_\nu + R_\nu w_z - R_zw_\nu = 0, \tag{1.10}$$

where primes at the variables t' and z' are omitted for convenience. These equations are supplemented by the initial conditions

$$w(0,z,\nu) = w_0(z,\nu), \qquad h(0,z,\nu) = h_0(z,\nu), \qquad R(0,z,\nu) = R_0(z,\nu). \tag{1.11}$$

Here from the requirement of one-to-oneness of the substitution of independent variables performed, the function $R_0(z,\nu)$ is considered a monotonically increasing function of the argument ν.

We write system (1.10) in a more illustrative form. For this, we integrate the first relation over the variable ν in the limits from ν to 1, and then, using boundary conditions (1.6), we eliminate the modified dimensionless pressure field p_* from it and substitute it into the second equation of the same system of relations. As a result, we obtain the equation

$$w_t + ww_z = \frac{æ^2 R_{1z}}{2R_1^2} - \frac{(h_1^2 R_1)_z}{2} + \frac{(h^2)_z R}{2} + \frac{1}{2}\Big(\int\limits_{\nu}^{1} R(h^2)_{\nu_1}\,d\nu_1\Big)_z, \tag{1.12}$$

where h_1 and R_1 are the values of the functions h and R, respectively, on the free surface of the jet $\nu = 1$, and, by virtue of the second boundary condition of system (1.9), $R_1(t,z) \equiv \eta_1(t,z)$.

In the last equation of system (1.10), the function q is substituted by its corresponding expression (1.8), as a result of which, this equation becomes

$$(R_\nu)_t + (wR_\nu)_z = 0. \tag{1.13}$$

Below it is assumed that the azimuthal component of the magnetic field inside the conducting jet is directly proportional to the radial coordinate: $h \equiv h_1 = \text{const}$ [3]. This assumption, on the one hand, reduces the third relation of system (1.10) to an identity; on the other hand, it leads to a marked simplification of Eq. (1.12), which now can be written as

$$w_t + ww_z = [(æ/R_1)^2 - h_1^2]R_{1z}/2. \tag{1.14}$$

The initial conditions for relations (1.13) and (1.14) are conditions (1.11) for the functions R and w:

$$R(0,z,\nu) = R_0(z,\nu), \qquad w(0,z,\nu) = w_0(z,\nu). \tag{1.15}$$

It should be noted that equations similar to relations (1.13) and (1.14) can also be derived in the case where R is considered a monotonically decreasing function of the argument ν. In this case, unlike in the case considered above, the role of the free boundary of the jet is played by the straight line $\nu = 0$ and the role of the symmetry axis is played the straight line $\nu = 1$.

The initial-boundary-value problem (1.13)–(1.15) has the energy integral

$$E_1 \equiv \frac{1}{2}\int\limits_{-\infty}^{+\infty}\Big(\int\limits_{0}^{1} w^2 R_\nu\,d\nu + æ^2 \ln R_1 + \frac{h_1^2}{2}\,R_1^2\Big)\,dz = \text{const} \tag{1.16}$$

under the assumption that the solutions of this problem are periodic along the z axis or are localized on it (in the last case, the fluid flow at infinity is homogeneous along the z coordinate).

It is easy to show that the mixed problem (1.13)–(1.15) has one more integral of motion. That is, differentiating (1.14) over the independent variable ν, we obtain

$$w_{\nu t} + (ww_z)_\nu = 0. \tag{1.17}$$

Next, from Eqs. (1.13) and (1.17) follows the important relation

$$C_t + wC_z = 0 \tag{1.18}$$

($C \equiv R_\nu/w_\nu$), which, combined with (1.17), shows that the functional

$$I \equiv \int\limits_{-\infty}^{+\infty}\int\limits_{0}^{1} w_\nu F(C)\,d\nu\,dz \tag{1.19}$$

is the desired additional integral of motion [3, 4]; $F(C)$ is an arbitrary function.

The exact steady-state solutions of the initial-boundary-value problem (1.13)–(1.15) can be written as

$$w = w^0(\nu), \qquad R = R^0(\nu), \qquad R_1 = R_1^0 \equiv 1. \tag{1.20}$$

Here w^0 and R^0 are arbitrary and monotonically increasing functions of the independent variable ν; the steady-state radius of the conducting jet is set equal to its characteristic radius r_0. It is easy to verify that the functions w^0, R^0, and R_1^0 [see (1.20)] identically satisfy Eqs. (1.13) and (1.14).

The objective of the further study is to find conditions under which the steady-state flows (1.20) are stable to small axisymmetric long-wave perturbations $w'(t,z,\nu)$, $R'(t,z,\nu)$, and $R_1'(t,z)$.

2. Stability of Arbitrary Steady-State Axisymmetric Shear Jet Flows of an Ideal Fluid with a Free Surface in an Azimuthal Magnetic Field. The mixed problem (1.13)–(1.15) and relations (1.17) and (1.18) are linearized on the exact steady-state solutions (1.20). The linearization yields the initial-boundary-value problem

$$w'_t + w^0 w'_z = \frac{1}{2}\,(æ^2 - h_1^2)R'_{1z}, \qquad R'_{\nu t} + w^0 R'_{\nu z} + \frac{dR^0}{d\nu}\,w'_z = 0,$$

$$w'_{\nu t} + \frac{dw^0}{d\nu}\,w'_z + w^0 w'_{z\nu} = 0, \qquad C'_t + w^0 C'_z = 0,$$

$$C' \equiv \Big(\frac{dw^0}{d\nu}\Big)^{-1}[R'_\nu - C^0 w'_\nu], \qquad C^0 \equiv \frac{dR^0}{d\nu}\Big(\frac{dw^0}{d\nu}\Big)^{-1}, \tag{2.1}$$

$$w'(0,z,\nu) = w'_0(z,\nu), \qquad R'(0,z,\nu) = R'_0(z,\nu)$$

on whose solutions the functional

$$E \equiv \int\limits_{-\infty}^{+\infty}\int\limits_0^1 \Big[\frac{1}{2}\,\frac{dR^0}{d\nu}\,w'^2 + w^0 w' R'_\nu + \frac{1}{2}\,\frac{dw^0}{d\nu}\,\frac{d^2F}{dC^2}\,(C^0)C'^2\Big]\,d\nu\,dz + \frac{h_1^2 - æ^2}{4}\int\limits_{-\infty}^{+\infty} R_1'^2\,dz = \text{const.} \tag{2.2}$$

is conserved with time.

The first variation δJ_1 of the integral $J_1 \equiv E_1 + I = \text{const}$ [see (1.16) and (1.19)] vanishes on the steady-state flows (1.20) if the functions w^0, R^0, and F reduces the equation

$$\frac{dF}{dC}\,(C^0) = -\frac{w^{02}}{2}$$

to an identity and its second variation $\delta^2 J_1$, written in the corresponding notation, is similar in form to the functional E.

The exact steady-state solutions (1.20) of the mixed problems (1.13)–(1.15) are stable to small axisymmetric long-wave perturbations (2.1) if and only if the integral E in (2.2) is of fixed sign.

To determine whether the functional E possesses the property of having fixed sign, we write it as

$$E = \int\limits_{-\infty}^{+\infty}\int\limits_0^1 (A\boldsymbol{u},\boldsymbol{u})\,d\nu\,dz, \qquad \boldsymbol{u} \equiv (w', R'_\nu, C', R'_1)^{\mathrm{t}}, \tag{2.3}$$

where $A = \|a_{ik}\|$ is a 4×4 square matrix with nonzero elements:

$$a_{11} = \frac{1}{2}\,\frac{dR^0}{d\nu}, \quad a_{12} = a_{21} = \frac{w^0}{2}, \quad a_{24} = a_{42} = \frac{h_1^2 - æ^2}{8}, \quad a_{33} = \frac{1}{2}\,\frac{dw^0}{d\nu}\,\frac{d^2F}{dC^2}\,(C^0).$$

According to the Silvester criterion [5], the integrand of the functional E in (2.3) is positive (negative) definite if and only if the principal minors of the matrix A are positive [have a factor $(-1)^m$, where m is the order of one principal minor or another].

It is easy to show that the principal minors of the matrix A do not possess the required property of having fixed sign. Thus, for the positive definiteness of the integrand of the functional E, the following inequalities should hold:

$$\frac{dR^0}{d\nu} > 0, \qquad -w^{02} > 0.$$

It is obvious that the second inequality cannot hold in principle because the negative number is naturally less than zero. At the same time, for the negative definiteness of this integrand, it is necessary that the following inequalities be satisfied:

$$\frac{dR^0}{d\nu} < 0, \qquad -w^{02} > 0.$$

The first inequality is not satisfied because the function R^0 is monotonic, and the second inequality is not satisfied because the quantity w^{02} is positive.

Thus, the functional E [see (2.3)] is neither positive nor negative definite. This, in turn, implies the absence of sufficient conditions for the stability of the exact steady-solutions (1.20) of the initial-boundary-value problems (1.13)–(1.15) against small axisymmetric long-wave perturbations $w'(t,z,\nu)$, $R'(t,z,\nu)$, and $R_1'(t,z)$ [see (2.1)], which are understood as the conditions for the sign-definiteness of the energy integral of motion E. Obviously, a similar situation will be observed in the case of more general formulations of the problems of the stability of steady-state shear jet magnetohydrodynamic flows with a free boundary.

3. Stability of Partial Steady-State Axisymmetric Shear Jet Flows of an Ideal Fluid with a Free Surface in an Azimuthal Magnetic Field. Below, using the direct Lyapunov method [6, 7], we obtain the necessary and sufficient condition for the stability of the subclass of steady-state flows (1.20)

$$\frac{d}{d\nu}\,(w^0C^0) \leqslant 0 \tag{3.1}$$

against small axisymmetric long-wave perturbations (2.1) which do not change the value of the function $C^0(\nu)$ for each fluid particle and satisfy a number of constraints on the symmetry axis and free boundary of the examined jet.

To show that any exact steady-state solution (1.20), (3.1) of the mixed problem (1.13)–(1.15) is unstable to small axisymmetric long-wave perturbations $w'(t,z,\nu)$, $R'(t,z,\nu)$, $R_1'(t,z)$ [see (2.1)], it is necessary to find even one of these perturbations that increases exponentially in time. To this end, a study is made of the axisymmetric shear jet flows of a perfectly conducting inviscid incompressible fluid with a free surface in an azimuthal magnetic field for which the small perturbations $C'(t,z,\nu)$ [see (1.18) and (2.1)] are equal to zero. In other words, it is assumed that for any fluid particle, the value of the function C^0 [(see (1.20), (2.1), and (3.1)] does not change under the perturbations, i.e., these perturbations are the deviations of the fluid-particle trajectories from the corresponding streamlines of the steady-state flows (1.20) and (3.1).

From the physical point of view, the above requirement to small perturbations is justified by the fact that the velocity circulation over any fluid contour in the axis plane specified at the initial time is conserved during perturbation propagation because the value of the function C [see 1.18)] does not change in the fluid particles, according to the initial-boundary-value problem (1.13)–(1.15). These perturbations can be introduced using the Lagrangian displacement field $\xi = \xi(t,z,\nu)$ [8] defined by the equation

$$\xi_t = w' - w^0\xi_z. \tag{3.2}$$

In view of relations (3.2), the mixed problem (2.1) can be written as

$$\begin{gathered} w_t' + w^0w_z' = \frac{1}{2}\,(æ^2 - h_1^2)R_{1z}', \qquad R_\nu' = -\frac{dR^0}{d\nu}\,\xi_z, \qquad w_\nu' = -\frac{dw^0}{d\nu}\,\xi_z, \\ R_\nu' = C^0w_\nu', \qquad \xi(0,z,\nu) = \xi_0(z,\nu), \qquad w'(0,z,\nu) = w_0'(z,\nu). \end{gathered} \tag{3.3}$$

By direct calculations, the functional E in (2.2) is brought to the form

$$E = \frac{1}{2}\int\limits_{-\infty}^{+\infty}\Big[\int\limits_0^1 \frac{d}{d\nu}\,(R^0 - w^0C^0)w'^2\,d\nu + \frac{h_1^2 - æ^2}{2}\,R_1'^2\Big]\,dz \tag{3.4}$$

and serves as the integral of motion for the initial-boundary-value problem (3.2), (3.3) if the following equalities hold:

$$\begin{gathered} \int\limits_{-\infty}^{+\infty}(w^0C^0w'^2)\Big|_{\nu=1}\,dz = \int\limits_{-\infty}^{+\infty}(w^0C^0w'^2)\Big|_{\nu=0}\,dz, \\ \int\limits_{-\infty}^{+\infty}(w^0C^0w')\Big|_{\nu=1}R_{1z}'\,dz = \int\limits_{-\infty}^{+\infty}(w^0C^0w')\Big|_{\nu=0}R_{1z}'\,dz. \end{gathered} \tag{3.5}$$

It is important to note that if one of conditions (3.5) is satisfied, the other conditions is also satisfied. In addition, since the function $w'(t,z,\nu)$ [see (3.2) and (3.3)], as a function of the independent variable ν, has an arbitrariness on the symmetry axis $\nu = 0$ of the conducting jet and its free surface $\nu = 1$, these equalities can be interpreted as the boundary conditions of the mixed problem (3.2), (3.3).

An analysis of the expression for the functional E in (3.4) shows that if the inequality

$$h_1^2 \geqslant æ^2 \tag{3.6}$$

is valid, then, because the function R^0 is monotonic and because the integral E does not depend on time, this inequality implies the stability of the exact steady-state solutions (1.20) and (3.1) of the initial-boundary-value problems (1.13)–(1.15) against small axisymmetric long-wave perturbations $\xi(t, z, \nu)$ [see (3.2), (3.3), and (3.5)].

Let inequality (3.6) be violated; i.e., let the following relation be valid:

$$h_1^2 < æ^2. \tag{3.7}$$

Then, it is possible to show that the steady-state flows (1.20), (3.1) are unstable to the small axisymmetric long-wave perturbations (3.2), (3.3), and (3.5).

Indeed, doubly differentiating the auxiliary functional

$$M \equiv \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{1} \frac{dR^0}{d\nu}\, \xi^2 \, d\nu \, dz \tag{3.8}$$

over the independent variable t and using the constraints (3.2)–(3.5), we obtain the so-called virial equality [8–10]

$$\frac{d^2 M}{dt^2} = 4(T - \Pi), \tag{3.9}$$

where

$$T \equiv \frac{1}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{1} \frac{dR^0}{d\nu}\, w'^2 \, d\nu \, dz, \qquad \Pi \equiv \frac{h_1^2 - æ^2}{4} \int\limits_{-\infty}^{+\infty} R_1'^2 \, dz.$$

Multiplying equality (3.9) by a certain constant λ and taking into account the relation

$$E \equiv T + T_1 + \Pi = \text{const}, \tag{3.10}$$

where

$$T_1 \equiv -\frac{1}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{1} \frac{d}{d\nu} (w^0 C^0) w'^2 \, d\nu \, dz \geqslant 0,$$

we obtain the basic equation

$$\frac{dE_\lambda}{dt} = 2\lambda E_\lambda - 4\lambda T_\lambda - 2\lambda T_1, \tag{3.11}$$

where

$$E_\lambda \equiv \Pi_\lambda + T_\lambda, \qquad 2\Pi_\lambda \equiv 2(\Pi + T_1) + \lambda^2 M,$$

$$2T_\lambda \equiv 2T - \lambda \frac{dM}{dt} + \lambda^2 M = \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{1} \frac{dR^0}{d\nu} (w' - \lambda \xi)^2 \, d\nu \, dz \geqslant 0.$$

Because the quantities T_1 [see (3.10)] and T_λ are nonnegative, from relation (3.11) for $\lambda > 0$ we obtain the differential inequality

$$\frac{dE_\lambda}{dt} \leqslant 2\lambda E_\lambda,$$

whose integration yields the important estimate

$$E_\lambda(t) \leqslant E_\lambda(0) \exp(2\lambda t). \tag{3.12}$$

Relation (3.12) is true for any solutions of the mixed problem (3.2), (3.3), (3.5) and for arbitrary positive values of λ. In addition, in finding this inequality, we did not impose any restrictions on the sign of the functional Π in (3.9).

From relation (3.12) it follows that the integral E_λ, generally speaking, varies monotonically in time. This allows us to treat this functional as Lyapunov's functional [6, 7, 9, 10].

Next, using inequality (3.12), we shall construct two-sided exponential estimates for the growth of small axisymmetric long-wave perturbations $\xi(t, z, \nu)$ [see (3.2), (3.3), and (3.5)] and among the latter we shall choose and describe the most rapidly growing small perturbations.

Using relation (3.7) and choosing appropriate initial Lagrangian displacement fields $\xi_0(z, \nu)$ and velocity field perturbations $w_0'(z, \nu)$ [see (3.3)], it is easy to ensure the validity of the inequalities $\Pi(0) < 0$ and $T(0) + T_1(0) < |\Pi(0)|$. As a result, the integral $E_\lambda(0)$, as follows from (3.11), is a second-order polynomial of the parameter λ with a positive coefficient $M(0)$ [see (3.8)] at λ^2 and a negative free term $E(0)$ [see (3.4)]:

$$E_\lambda(0) = E(0) - \frac{\lambda}{2}\frac{dM}{dt}(0) + \lambda^2 M(0). \tag{3.13}$$

If the values of λ are taken from the interval

$$0 < \lambda < \Lambda \equiv A_1 + \sqrt{A_2}, \tag{3.14}$$

where

$$A_1 \equiv [4M(0)]^{-1}\frac{dM}{dt}(0), \qquad A_2 \equiv A_1^2 - \frac{E(0)}{M(0)},$$

then from relation (3.13) follows the estimate $E_\lambda(0) < 0$. This estimate and inequality (3.12) indicate that the small axisymmetric long-wave perturbations (3.2), (3.3), and (3.5) grow exponentially in time.

Provided that $\lambda \equiv \Lambda - \delta_1$ (with any parameter δ_1 from the interval $]\,0, \Lambda\,[\,$), relation (3.12) can be written as

$$E_{\Lambda-\delta_1}(t) \leqslant E_{\Lambda-\delta_1}(0)\exp[2(\Lambda - \delta_1)t] \qquad [E_{\Lambda-\delta_1}(0) < 0]. \tag{3.15}$$

Since, by virtue of the representation (3.11), the inequality $E_\lambda(t) > \Pi(t)$ is satisfied, relation (3.15) can be written as

$$-\Pi(t) > |E_{\Lambda-\delta_1}(0)|\exp[2(\Lambda - \delta_1)t]$$

or, finally,

$$(æ^2 - h_1^2)\int\limits_{-\infty}^{+\infty} R_1'^2\,dz > 4|E_{\Lambda-\delta_1}(0)|\exp[2(\Lambda - \delta_1)t]. \tag{3.16}$$

From inequality (3.16) it follows that the quantity $\Lambda - \delta_1$ [see (3.14) and (3.15)] is the lower bound of the increments of the small axisymmetric long-wave perturbations $\xi(t, z, \nu)$ [see (3.2), (3.3), and (3.5)].

Estimate (3.16) can be considerably improved if the initial Lagrangian displacement field $\xi_0(z, \nu)$ and the velocity field perturbation $w_0'(z, \nu)$ [see (3.3)] are additionally subjected to the requirement

$$w_0'(z, \nu) = \lambda\xi_0(z, \nu). \tag{3.17}$$

Indeed, from relations (3.11) and (3.13) it follows that $T_\lambda(0) = 0$ and $E_\lambda(0) = \Pi_\lambda(0)$. In turn, these equalities show that on the interval

$$0 < \lambda < \Lambda_1 \equiv \sqrt{-2\Pi(0)/M(0)} \tag{3.18}$$

the estimate $\Pi_\lambda(0) < 0$ is valid. From this it follows that setting $\lambda \equiv \Lambda_1 - \delta_2$ (with an arbitrary parameter δ_2 from the interval $]\,0, \Lambda_1[\,$), it is possible to bring inequality (3.12) to the form

$$E_{\Lambda_1-\delta_2}(t) \leqslant \Pi_{\Lambda_1-\delta_2}(0)\exp[2(\Lambda_1 - \delta_2)t] \qquad [\Pi_{\Lambda_1-\delta_2}(0) < 0]. \tag{3.19}$$

After calculations similar to those performed above in justifying estimate (3.16), inequality (3.19) becomes

$$-\Pi(t) > |\Pi_{\Lambda_1-\delta_2}(0)|\exp[2(\Lambda_1 - \delta_2)t]$$

or, finally,

$$(æ^2 - h_1^2)\int\limits_{-\infty}^{+\infty} R_1'^2\,dz > 4|\Pi_{\Lambda_1-\delta_2}(0)|\exp[2(\Lambda_1 - \delta_2)t]. \tag{3.20}$$

According to relation (3.20), the quantity $\Lambda_1 - \delta_2$ [see (3.18) and (3.19)] is the lower bound of the increments of the small axisymmetric long-wave perturbations (3.2), (3.3), (3.5), and (3.17).

From a comparison of inequalities (3.16) and (3.20), it follows that the small axisymmetric long-wave perturbations $\xi(t, z, \nu)$ [see (3.2), (3.3), and (3.5)] whose initial data satisfy constraint (3.17) grow faster than the remaining perturbations of the examined subclass, and among them, the most rapidly growing perturbations are those whose increments, as shown below, are calculated by the formula

$$\Lambda_1^+ \equiv \sup_{\xi_0(z,\nu)} \Lambda_1. \tag{3.21}$$

Indeed, let $\lambda > \Lambda_1^+$. In this case, for all possible initial Lagrangian displacement fields $\xi_0(z,\nu)$ [see (3.3)], the relation $\Pi_\lambda(0) > 0$ is valid. Hence, the integral $E_\lambda(0)$ [see (3.11) and (3.13)] is also positive definite for all admissible initial Lagrangian displacement fields $\xi_0(z,\nu)$ and velocity field perturbations $w_0'(z,\nu)$ [see(3.3)].

For $\lambda \equiv \Lambda_1^+ + \delta_3$ ($\delta_3 > 0$ is a parameter) from inequality (3.12) follows the estimate

$$E_{\Lambda_1^+ + \delta_3}(t) \leqslant E_{\Lambda_1^+ + \delta_3}(0) \exp{[2(\Lambda_1^+ + \delta_3)t]}. \tag{3.22}$$

By virtue of (3.22), the quantity $\Lambda_1^+ + \delta_3$ is the upper bound of the increments of the small axisymmetric long-wave perturbations (3.2), (3.3), and (3.5).

A comparison of inequalities (3.20) and (3.22) leads to the conclusion that the quantity Λ_1^+ [see (3.18) and (3.21)] is both the upper and lower bound of the growth rate ω of the small perturbations (3.2), (3.3), and (3.5):

$$\Lambda_1^+ - \delta_2 \leqslant \omega \leqslant \Lambda_1^+ + \delta_3. \tag{3.23}$$

Relation (3.23) shows that the higher growth rate is observed for small axisymmetric long-wave perturbations $\xi(t,z,\nu)$ [see (3.2), (3.3), and (3.5)] whose increment is close in magnitude to Λ_1^+.

Thus, if condition (3.7) is valid, then determining, with the use of relations (3.18) and (3.21), the value of Λ_1^+, which estimates the growth rate ω [see (3.23)] for the most rapidly growing small perturbations (3.2), (3.3), (3.5), and (3.17), it is possible to find the characteristic times during which the small axisymmetric long-wave perturbations (3.2), (3.3), and (3.5) cause failure of the steady-state axisymmetric shear jet flows (1.20) and (3.1) of a perfectly conducting inviscid incompressible fluid with a free surface in an azimuthal magnetic field.

Next, we construct an example of the steady-state flows (1.20), (3.1) and the initial small axisymmetric long-wave perturbations (3.2), (3.3), (3.5) imposed on them, which, generally speaking, evolve with time according to estimates (3.16) and (3.22). A study is made of the steady-state axisymmetric shear jet magnetohydrodynamic flows

$$w^0(\nu) = C_1 \exp{(-C_2\nu)}, \qquad R^0(\nu) = \nu, \qquad R_1^0 = 1 \tag{3.24}$$

(C_1 and C_2 are positive constants) of an ideal fluid in the infinite strip

$$\Big[(z,\nu)\colon\ -\infty < z < +\infty,\ 0 \leqslant \nu \leqslant 1\Big]. \tag{3.25}$$

It is easy to show that these flows are typical representatives of the particular class (3.1) of the steady-state flows (1.20).

If inequality (3.7) is satisfied, the steady-state flows (3.24) and (3.25) are unstable, for example, to small axisymmetric long-wave perturbations $\xi(t,z,\nu)$ [see (3.2), (3.3), and (3.5)] for which the initial Lagrangian displacement field $\xi_0(z,\nu)$ is specified as

$$\xi_0(z,\nu) = (2\nu - 1)\exp{(C_2\nu)}\sin{(2\pi z/l)}, \tag{3.26}$$

where l is an arbitrary positive constant. From the physical point of view, these perturbations are periodic (with a wavelength l) fluctuations of the free boundary of the jet and the axial velocity of the fluid flowing inside the jet.

Indeed, using the definition of the function $R_1(t,z)$ [see (1.9) and (1.12)] and Eq. (3.3), it is easy to obtain the relations

$$R_{0\nu}'(z,\nu) = -(2\pi/l)(2\nu - 1)\exp{(C_2\nu)}\cos{(2\pi z/l)},$$

$$R_1'(0,z) \equiv \int_0^1 R_{0\nu}'(z,\nu)\,d\nu = -\frac{2\pi}{lC_2}\Big[\Big(1 - \frac{2}{C_2}\Big)\exp{(C_2)} + \frac{2}{C_2} + 1\Big]\cos{\frac{2\pi z}{l}},$$

$$w_{0\nu}'(z,\nu) = (2\pi C_1 C_2/l)(2\nu - 1)\cos{(2\pi z/l)}, \qquad w_0'(z,\nu) = \int_0^\nu w_{0\nu_1}'(z,\nu_1)\,d\nu_1 = \frac{2\pi C_1 C_2}{l}\,(\nu^2 - \nu)\cos{\frac{2\pi z}{l}}.$$

It should be noted that since here

$$w_0'(z,0) = 0, \qquad w_0'(z,1) = 0, \tag{3.27}$$

boundary conditions (3.5) are identically satisfied, and, hence, at $t = 0$, they are matched to initial conditions (3.3) and (3.27).

Taking into account the periodicity of the field $\xi_0(z,\nu)$ [see (3.26)] over the independent variable z and expressions (3.9) and (3.10) for the functionals T, T_1, and Π, it is possible to calculate the values of the latter at the initial time:

$$T(0) \equiv \frac{1}{2}\int_0^l\int_0^1 \frac{dR^0}{d\nu}\, w_0'^2(z,\nu)\,d\nu\,dz = \frac{\pi^2 C_1^2 C_2^2}{30l}, \qquad T_1(0) \equiv -\frac{1}{2}\int_0^l\int_0^1 \frac{d}{d\nu}(w^0C^0)w_0'^2(z,\nu)\,d\nu\,dz = 0,$$

$$\Pi(0) \equiv \frac{h_1^2 - æ^2}{4}\int_0^l R_1'^2(0,z)\,dz = \frac{\pi^2(h_1^2 - æ^2)}{2lC_2^2}\Big[\Big(1 - \frac{2}{C_2}\Big)\exp(C_2) + \frac{2}{C_2} + 1\Big]^2.$$

From this it follows that the inequality $\Pi(0) < 0$ is true and the inequality $T(0) + T_1(0) < |\Pi(0)|$ will be valid if the constants C_1 and C_2 are chosen properly, for example: $0 < C_1 < (3 - e)\sqrt{15(æ^2 - h_1^2)}$, and $C_2 = 1$.

As a result, for the steady-state flows (3.24) and (3.25), we obtain the explicit forms of the lower bound (3.16) and the upper bound (3.22) (the second bound contains the parameter Λ_1 instead of Λ_1^+), which characterize the growth of the small axisymmetric long-wave perturbations (3.2), (3.3), (3.5), and (3.26), and, hence the instability of these flows. It should be noted that the growth rate ω in (3.23) for the small perturbations (3.2), (3.3), (3.5), and (3.26) is evaluated both from below and from above by the quantity Λ_1 (3.18) rather than Λ_1^+ (3.21).

Finally, the most rapidly growing small axisymmetric long-wave perturbations of the steady-state flows (3.24) and (3.25) are those for which the initial Lagrangian displacement field has the form $\xi_0(z,\nu) = f(w^0 - \lambda z)$ by virtue of Eqs. (3.3) and equality (3.17). In this case, the function f should be either periodic or localized on the z coordinate. Then, the nature of the growth of these perturbations can be judged from the lower bound (3.20) and the upper bound (3.22), and their growth rate ω [see (3.23)] can be found using the quantity Λ_1^+ [see (3.18) and (3.21)].

4. Instability of Arbitrary Steady-Sate Axisymmetric Shear Jet Flows of an Ideal Fluid with a Free Surface in an Azimuthal Magnetic Field. Below, using the direct Lyapunov method, we show that inequality (3.7) is a sufficient condition for the instability of the exact steady-state solutions (1.20) of the initial-boundary-value problem (1.13)–(1.15) against small axisymmetric long-wave perturbations $w'(t,z,\nu)$, $R'(t,z,\nu)$, and $R_1'(t,z)$ [see (2.1)], and the perturbation growing exponentially in time is sought among the representatives of the subclass (3.2), (3.3) of these perturbations. This implies that the steady-state flows (1.20) are further considered free from constraint (3.1), and the small perturbations (3.2), (3.3) are considered free from constraint (3.5).

According to the above assumptions, the integral E in (2.2), which, naturally, is conserved not only on the solutions of the mixed problem (2.1) but also on the solutions of the initial-boundary-value problem (3.2), (3.3), has the form (3.10) with the only exception that T_1 now denotes the functional

$$T_1 \equiv \int_{-\infty}^{+\infty}\int_0^1 w^0 w' R_\nu'\,d\nu\,dz \tag{4.1}$$

[it is important that the integral T_1 in the form (4.1) is no longer of fixed sign]. The virial equality (3.9) and the basic equation (3.11) also remain valid in this case.

In view of the aforesaid and under the assumption that condition (3.7) is valid, relation (3.11) leads to the differential inequality

$$\frac{d^2M}{dt^2} - 2\lambda\frac{dM}{dt} + 2\lambda^2 M \geqslant 0 \tag{4.2}$$

(λ is a certain positive constant). Integration of this inequality, in turn, results in the following lower bound:

$$M(t) \geqslant (C_3\cos\lambda t + C_4\sin\lambda t)\exp(\lambda t) \tag{4.3}$$

(C_3 and C_4 are known constants).

Because the functional M [see (3.8)] is nonnegative by the definition and because the right side of inequality (4.3) contains trigonometric functions bounded in absolute value, relation (4.3) can be brought, without loss of generality, to the form

$$M(t) \geqslant C_5\exp(\lambda t) \tag{4.4}$$

(C_5 is a known positive constant).

Relation (4.4) shows that the small axisymmetric long-wave perturbations $\xi(t,z,\nu)$ [see (3.2) and (3.3)] of the exact steady-state solutions (1.20) of the mixed problem (1.13)–(1.15) grow exponentially with time. It should be noted that the parameter λ in inequality (4.4) is not subjected to any additional constraints. In this sense, the detected instability can be interpreted as a peculiar "breakthrough" of small-scale perturbations (which were previously eliminated from consideration by passage to the long-wave approximation) to the region of large-scale fluid flows.

Next, we construct an example of the steady-state flows (1.20) and the small axisymmetric long-wave perturbations imposed on them (3.2), (3.3), which develop in time according to the lower bound (4.4) if relation (3.7) is satisfied.

A study is made of the steady-state axisymmetric shear jet magnetohydrodynamic flows of an ideal fluid

$$w^0(\nu) = a - \nu, \qquad R^0(\nu) = \nu, \qquad R_1^0 = 1 \tag{4.5}$$

($a > 1$ is a constant) in the infinite strip (3.25). Obviously, these flows belong to the class of steady-state flows (1.20).

If inequality (3.7) is true, the steady-state flows (4.5), (3.25) are unstable to small axisymmetric long-wave perturbations $\xi(t,z,\nu)$ [see (3.2), (3.3)] of the form

$$\xi(t,z,\nu) = \alpha \exp(\sigma\beta t)\Big[(\sigma^2 - (1/2-\nu)^2)(\cos\gamma\beta t\cos\beta z - \sin\gamma\beta t\sin\beta z)$$
$$+ 2\sigma(1/2-\nu)(\cos\gamma\beta t\sin\beta z + \sin\gamma\beta t\cos\beta z)\Big]/[\sigma^2 + (1/2-\nu)^2]^2. \tag{4.6}$$

Here α is an arbitrary constant and β is a positive constant, whereas $\sigma \equiv \sqrt{(æ^2 - h_1^2)/2 - 1/4}$, $\gamma \equiv 1/2 - a$.

Direct verification shows that the function $\xi(t,z,\nu)$ in (4.6) is a solution of the initial-boundary-value problem (3.2), (3.3) and satisfies relations (3.9) and (3.10) [with the integral T_1 in the form (4.1)] and (4.4). In addition, it can be used as a Hadamard example [11] because a certain initial arbitrariness is admitted in the choice of β in the exponent on the right of expression (4.6).

In view of the universality of the differential inequality (4.2), it may be effectively applied to a consideration of a wide range of problems of hydrodynamic stability.

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REFERENCES

1. R. V. Polovin and V. P. Demutskii, *Fundamentals of Magnetohydrodynamics* [in Russian], Énergoatomizdat, Moscow (1987).
2. V. E. Zakharov, "Benney equations and the quasi-classical approximation in the inverse problem method," *Funkts. Anal. Prilozh.*, **14**, No. 2, 15–24 (1980).
3. Yu. G. Gubarev and V. V. Nikulin, "Linear long-wave instability of one class of steady-state jet flows of an ideal fluid in the intrinsic electric current field," *Izv. Ross. Akad. Nauk., Mekh. Zhidk, Gaza*, No. 2, 64–75 (2001).
4. Yu. G. Gubarev, "On the analogy between the Benney equations and the Vlasov–Poisson equations," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 110, Novosibirsk (1995), pp. 78–90.
5. V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients and Their Applications* [in Russian], Nauka, Moscow (1972).
6. A. M. Lyapunov, *General Problem of Dynamic Stability* [in Russian], Gostekhteoretizdat, Moscow–Leningrad (1950).
7. N. G. Chetaev, *Dynamic Stability. Papers on Analytical Mechanics* [in Russian], Izd. Akad. Nauk SSSR, Moscow (1962).
8. S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, London Univ. Press, New Haven (1969).
9. Yu. G. Gubarev, "Instability of rotationally symmetric MHD flows," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 6, 19–25 (1995).
10. Yu. G. Gubarev, "On the instability of helical magnetohydrodynamic flows," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 1, 150–156 (1999).
11. S. K. Godunov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1979).